

A Serre Derivative for even weight Jacobi Forms

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Abstract

In this paper so called deformed Eisenstein Series are introduced and studied. In particular, their modular and periodic properties are considered and a completion to meromorphic Jacobi forms of index 0 given. It is shown that there is a explicit formula of these functions in terms of the classical theta function θ_1 and the same construction is considered for $\theta_2, \theta_3, \theta_4$. Afterwards it is explained how to use the almost Jacobi Forms behaviour of deformed Eisenstein Series to write down differential operators for (possibly weak, even weight) Jacobi Forms. In particular we will give a direct generalization of the classical Serre Derivative to even weight Jacobi Forms. These operators can be used to study differential equations for Jacobi forms. As a result, we will apply the generalized Serre Derivative to obtain Ramanujan-style equations for $E_{4,1}, E_{6,1}$ and a newly defined $E_{2,1}$.

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0 Introduction

Let E_2, E_4, E_6 be the classical Eisenstein Series of weight 2, 4, 6 and recall the Serre derivative $\partial^{SD} f = f' - \frac{k}{12} E_2 f$ where k is the weight of the quasi modular form f . A bit less than 100 years ago, Ramanujan proved in 1916 [Ram00] the following set of equations

$$\begin{aligned}\partial^{SD}(E_2) + \frac{1}{12}E_2^2 &= -\frac{1}{12}E_4 \\ \partial^{SD}(E_4) &= -\frac{1}{3}E_6 \\ \partial^{SD}(E_6) &= -\frac{1}{2}E_4^2\end{aligned}\tag{1}$$

In 1985 Eichler and Zagier [EZ85] introduced and studied Jacobi Forms, which are 2 variable generalizations of classical Modular Forms. These functions transform in one variable like a modular form and in the other like an elliptic function. As for modular forms, one can define Eisenstein Series for Jacobi Forms. The most useful ones are the Eisenstein Series of index 1 and weight 4 and 6, denoted $E_{4,1}$ and $E_{6,1}$. When setting the elliptic variable in a Jacobi Form of weight k and index m equal to zero, one obtains a honest modular form of weight k . In particular one has $E_{4,1}(0, \tau) = E_4(\tau)$ and $E_{6,1}(0, \tau) = E_6(\tau)$.

In this paper we introduce a generalized Serre Derivative, denoted by ∂^J , for even weight Jacobi Forms. This derivative will restrict to the classical Serre derivative when setting the elliptic variable equal to zero. Further we introduce a function $E_{2,1}$ analogous to E_2 , that like his modular counterpart transforms almost like a Jacobi Form but adds an extra term under the modular equation. Using these two new ingredients, we will then prove for index 1 Jacobi Forms an analog of Ramanujan's original equations. The equations read:

$$\begin{aligned}\partial^J E_{2,1} + \frac{1}{12}E_2 E_{2,1} + \frac{1}{16}E_4' \phi_{-2,1} &= -\frac{1}{12}E_{4,1} \\ \partial^J E_{4,1} &= -\frac{1}{3}E_{6,1} \\ \partial^J E_{6,1} &= -\frac{1}{2}E_4 E_{4,1}\end{aligned}$$

where $\phi_{-2,1}$ is the weak Jacobi form found in [EZ85]. This is a direct generalization of Ramanujan's equations as $\phi_{-2,1}$ vanishes for $z = 0$.

The new input in this paper is a study of what physicists first called twisted Eisenstein Series. These are functions J_n defined by

$$J_n(z, \tau) = \delta_{n,1} \frac{p}{p-1} + B_n - n \sum_{k,r \geq 1} r^{n-1} (p^k + (-1)^n p^{-k}) q^{kr}$$

for $n \geq 0$ and $p = e^{2\pi iz}, z \in \mathbb{C}, q = e^{2\pi i\tau}, \tau \in \mathbb{H}$. As the term twisted Eisenstein Series is already used for Eisenstein Series twisted by a Dirichlet character, we will call the functions J_n *deformed* Eisenstein Series. They were considered for the first time in papers on $N = 2$ superconformal field theories where one was led naturally to a definition of those functions to obtain differential equations for elliptic genera (which are vector valued weak Jacobi Forms).

See in particular [GK09] and [MTZ08] for more information. In [GK09] Gaberdiel and Keller study the modular and periodic properties of deformed Eisenstein Series. By using arguments coming from conformal field theory they obtain a set differential operators for weak Jacobi forms.

In contrast, the main motivation for the author to study deformed Eisenstein Series and differential equation for Jacobi forms lies in Gromov Witten Theory. Given a smooth projective variety, one can define Gromov Witten Invariants and write the genus 0 invariants into a generating function Φ , the so called genus 0 Gromov Witten Potential. A main tool in studying the GW Invariants is the WDVV equation which gives a systems of differential equations for the function Φ . For a certain 4-fold, this procedure naturally leads to deformed Eisenstein series, Jacobi Forms and differential equations between them. In trying to understand these functions and differential equations, the author was led to this field of study. For more information it is referred to [CK99], [HKK⁺03] and [Obe].

In this paper we avoid any physics or Gromov Witten background and start with the above definition for the deformed Eisenstein Series. This definition differs by a multiple from the ones given in [GK09] and [MTZ08] - this results into more readable formulas along the way. As in [GK09], we derive formulas for the behaviour of J_n under the modular $((z, \tau) \mapsto (z/\tau, -1/\tau))$ and the elliptic $((z, \tau) \mapsto (z + \lambda\tau + \mu, \tau))$ transformation. This follows mainly the argument in [GK09]. We then define functions K_n by

$$K_n = \sum_{k=0}^n (-1)^{n+k} \binom{n}{k} J_k J_1^{n-k}$$

This are meromorphic, double periodic Jacobi Forms of index 0 with poles of order n at $z = 0$ and nowhere else in a fundamental region. A similar but not equal set of functions as K_n are the classical two variable Eisenstein Series, see e.g. [Wei99] and [Lib11]. Holomorphic double periodic functions are constants. Hence when taking τ as a parameter fixed and expanding K_n in a Laurent expansion around $z = 0$, the principal and constant part of K_n encodes all the information of the function. In the case of K_n , the coefficients of the Laurent series will be modular forms and so for a fixed n , K_n will live in the finitely generated free M_* module \mathbb{V}_n . Here M_* is the ring of holomorphic modular forms and \mathbb{V}_n will be defined as follows. We first define \mathbb{V} as the space of all meromorphic Jacobi forms of index 0, that have a pole in a fundamental region only at $z = 0$ and have modular forms as coefficients for the Laurent expansion at $z = 0$. Then we set

$$\mathbb{V}_n = \{f \in \mathbb{V} \mid \text{order of the pole is } \leq n\}$$

With the help of deformed Eisenstein Series we can then easily write down two differential operators for functions in \mathbb{V} . In particular, these operators will act on K_n . Since the vector space of modular forms is well understood, we can then derive relations between K_n and products and derivatives of K_n .

These relations are nothing especially new. Indeed the Weierstrass \wp function shares the same properties as K_2 and we have $\wp(z) = 4\pi^2 K_2(z, \tau)$. Another connection to classical functions is given by $J_1(z, \tau) = \theta_1^\bullet(z, \tau)/\theta_1(z, \tau)$ where $f^\bullet = \frac{1}{2\pi i} \frac{\partial}{\partial z} f$. We show that this is only the first in a series of identities relating the J_n to the classical theta function θ_1 . When we define the generating function

$$\mathcal{J} = \sum_{n \geq 0} \frac{J_n(z, \tau)}{n!} x^n$$

for some variable x , then

$$\mathcal{J} = x \frac{\theta_1^\bullet(0, \tau)}{\theta_1(\frac{x}{2\pi i})} \frac{\exp(x\partial_p) \cdot \theta_1(z, \tau)}{\theta_1(z, \tau)}$$

with $\partial_p f = f^\bullet$ and $\exp(x\partial_p) = \sum_{k \geq 0} x^k \partial_p^k$. A differential equation between the J_n then leads to the formula

$$\mathcal{J}' = \left(\frac{\partial}{\partial x} - \frac{1}{x} \right) \mathcal{J}^\bullet$$

with $f' = \frac{1}{2\pi i}(\partial/\partial\tau)f$. When written out this will lead to an infinite sequence of differential equations for θ_1 . We show afterwards that a similar construction and equations as above also hold for the other theta functions and give corresponding versions of deformed Eisenstein Series, called $J_{2,n}, J_{3,n}, J_{4,n}$.

Let finally $\phi = \phi_{-2,1}$ be the classical weak Jacobi form of weight -2 , index 1 . Given a weak Jacobi Form F of index m , we consider $\frac{F}{\phi^m}$. We show that this lies in \mathbb{V}_{2m} and by exploiting this connection we can study differential operators and equations for (weak) Jacobi forms in \mathbb{V} . Since we understand \mathbb{V} relatively well, we can recover the operators found in [GK09] with this idea. When one is looking out for operators defined just for even weight Jacobi forms, one can find more operators than in [GK09] and this will eventually lead to the formula for ∂^J given by

$$\partial^J F = F' - \frac{k}{12} E_2 F + \frac{1}{1-4m} \left(F^{\bullet\bullet} - J_1 F^\bullet + m J_2 F - \frac{m}{6} E_2 F \right)$$

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1 Notation

Our notational conventions for this paper are as follows:

- We use the following variables throughout:

$$z \in \mathbb{C}, \quad \tau \in \mathbb{H}, \quad w = 2\pi iz, \quad p = e^{2\pi iz}, \quad q = e^{2\pi i\tau}$$

- A function of the variables (z, τ) will be abbreviated as $J(z, \tau) = J(z) = J$, so for example $J(0)$ shall mean $J(0, \tau)$. A function of just τ can be abbreviated $E_2(\tau) = E_2$.
- We denote with \bullet and $'$ differentiation with respect to z and τ , or more precisely:

$$f^\bullet = \frac{1}{2\pi i} \frac{\partial}{\partial z} f = p \frac{\partial}{\partial p} f, \quad f' = \frac{1}{2\pi i} \frac{\partial}{\partial \tau} f = q \frac{\partial}{\partial q} f$$

- Our Bernoulli numbers obey the convention $B_1 = -\frac{1}{2}$ that is they are given as

$$B_k = 1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, -\frac{1}{30}, 0, \frac{5}{66}, \dots$$

- We will use the definitions for Jacobi Forms taken from [EZ85]. M_* will denote the ring of holomorphic modular forms. The space of (weak) Jacobi forms of weight k , index m will be denote by $J_{k,m}$ ($\tilde{J}_{k,m}$). A meromorphic function $f : \mathbb{C} \times \mathbb{H} \rightarrow \mathbb{C} \cup \{\infty\}$ will be called a meromorphic Jacobi Form of weight k and index m when the elliptic and modular transformation equation for Jacobi Forms are satisfied.
- In Section 3.4 we define functions $J_{i,n}$. This is the only place where they appear and they are not important for the rest of the paper.
- For a function of two variables (z, τ) , the fundamental region will mean the set

$$\left\{ z = \lambda + \mu\tau \in \mathbb{C} \mid 0 \leq \lambda, \mu < 1 \right\}.$$

2 J

2.1 Definition

We like to look at functions of the form

$$\sum_{(k,r) \neq (0,0)} \frac{p^k}{(k\tau + r)^n}$$

with $n \geq 1$, where for $n = 1, 2$ we use the convention to sum first over k and then over r and to define $\sum_{r \in \mathbb{Z}} = \lim_{N \rightarrow \infty} \sum_{r=-N}^N$. Going along the lines of page 16 in [BvdGHZ08] we find the following Fourier expansions

$$\begin{aligned} \sum_{(k,r) \neq (0,0)} \frac{p^k}{(k\tau + r)^n} &= \sum_{r \neq 0} \frac{1}{r^n} + \frac{(-2\pi i)^n}{(n-1)!} \sum_{k,r \geq 1} r^{n-1} (p^k + (-1)^n p^{-k}) q^{kr} \\ &= \begin{cases} \frac{(-2\pi i)^{2g}}{(2g-1)!} \left(-\frac{B_{2g}}{2g} + \sum_{k,r \geq 1} r^{2g-1} (p^k + p^{-k}) q^{kr} \right) & \text{for } n = 2g \text{ even} \\ \frac{(-2\pi i)^{2g+1}}{(2g)!} \sum_{k,r \geq 1} r^{2g} (p^k - p^{-k}) q^{kr} & \text{for } n = 2g + 1 \text{ odd} \end{cases} \end{aligned}$$

when $n \geq 2$. For $n = 1$ a *formal* computation shows

$$\begin{aligned} \sum_{(k,r) \neq (0,0)} \frac{p^k}{k\tau + r} &= \sum_{k \neq 0} p^k \left(\sum_{r \in \mathbb{Z}} \frac{1}{k\tau + r} \right) \\ &= \sum_{k \geq 1} (p^k - p^{-k}) \frac{\pi}{\tan(k\pi\tau)} \\ &= (-2\pi i) \sum_{k \geq 1} (p^k - p^{-k}) \left(\frac{1}{2} + \sum_{r \geq 1} q^{kr} \right) \\ &= (-2\pi i) \left(\frac{-p}{p-1} + \frac{1}{2} + \sum_{k,r \geq 1} (p^k - p^{-k}) q^{kr} \right) \end{aligned}$$

where *formal* concerns the following step:

$$\frac{p}{p-1} = \frac{1}{2} \left(\frac{p}{p-1} + \frac{1}{1-p^{-1}} \right) = \frac{1}{2} - \frac{1}{2} \sum_{k \geq 1} (p^k - p^{-k})$$

Definition 1. The deformed Eisenstein Series J_n are to be defined by

$$J_n(z, \tau) = \delta_{n,1} \frac{p}{p-1} + B_n - n \sum_{k,r \geq 1} r^{n-1} (p^k + (-1)^n p^{-k}) q^{kr}$$

for $n \geq 0$.

Note that we define $J_0 \equiv 1$. From the above we have at least formally

$$J_n(z, \tau) = -\frac{n!}{(-2\pi i)^n} \sum_{(k,n) \neq (0,0)} \frac{p^k}{(k\tau + n)^M} \quad (n \geq 1)$$

A few properties are immediate. We have

$$J_{2g}(0, \tau) = B_{2g} E_{2g}(\tau), \quad J_{2g+1}(0, \tau) = 0$$

for $g \geq 1$, where E_{2g} are the classical Eisenstein Series. Also easy to see for $k \geq 1$ is

$$\frac{k}{k+1} J_{k+1}^\bullet = J'_k \quad (2)$$

2.2 Periodicity

Taking the Fourier expansion of our J -Functions and replacing p with pq^λ we find after a lengthy but easy calculation for $\lambda, \mu \in \mathbb{Z}$

$$J_n(z + \lambda\tau + \mu, \tau) = \sum_{k=0}^n (-1)^{n+k} \binom{n}{k} \lambda^{n-k} J_k(z, \tau)$$

We can combine products of the J -Functions to make them double periodic, namely lets define recursively the functions K_n as:

$$K_n = J_n - J_1^n - \sum_{q=2}^{n-1} \binom{n}{q} K_q J_1^{n-q} \quad (3)$$

with $n \geq 2$ and the sum empty for $n = 2$. We get for example

$$\begin{aligned} K_2 &= J_2 - J_1^2 \\ K_3 &= J_3 - 3J_2 J_1 + 2J_1^3 \end{aligned}$$

An explicit formula can be given by

$$K_n = \sum_{k=0}^n (-1)^{n+k} \binom{n}{k} J_k J_1^{n-k}$$

Proposition 1. K_i are double-periodic in z , that is

$$K_i(z + \lambda\tau + \mu, \tau) = K_i(z, \tau)$$

Proof. By induction on n and a calculation using equation (3). □

2.3 Modularity

We like to determine how the J -Functions behave under the modular transformation $(z, \tau) \mapsto (z/\tau, -1/\tau)$. For this consider the Fourier expansion of J_1 and plugging in the Taylor expansion for p we get the following expression for J_1 :

$$\begin{aligned} J_1(z, \tau) &= \frac{1}{2\pi iz} + \sum_{n \geq 1} \frac{(2\pi iz)^{2n-1}}{(2n-1)!} \cdot \left(\frac{B_{2n}}{2n} E_{2n}(\tau) \right) \\ &= \frac{1}{w} + \sum_{n \geq 1} \frac{w^{2n-1}}{(2n)!} \cdot \left(B_{2n} E_{2n}(\tau) \right) \end{aligned}$$

where $w = 2\pi iz$. Using this equation together with the transformation property of the Eisenstein series, we find:

$$J_1(z/\tau, -1/\tau) = z + \tau J_1(z, \tau)$$

In a similar fashion we could get the behaviour of the higher J_i function, but we use a different path. Suppose we know how $J_i(z, \tau)$ transforms. Then, by differentiating the transformation equation for J_i with respect to τ , and using the equation (2) we get an expression for $J_{i+1}^\bullet(z/\tau, -1/\tau)$. Integrating with respect to z , we find an expression for $J_{i+1}(z/\tau, -1/\tau)$ up to a function that depends only on τ . Plugging in $z = 0$ and using that J_{2g} restricts to standard Eisenstein series, for which we know the transformation property, while J_{2g+1} restricts to 0, we get an expression for J_{i+1} . In short we have:

Proposition 2 ([GK09]). $J_n(z/\tau, -1/\tau) = \sum_{k=0}^n \binom{n}{k} z^{n-k} \tau^k J_k$

Remark 2. From the Taylor expansion we can see that J_1 is not something particularly new. Indeed the Weierstrass \wp -function has the Taylor expansion

$$\wp(z) = \frac{1}{z^2} + \sum_{n \geq 1} (2n+1)2\zeta(2n+2)E_{2n+2}z^{2n}$$

and so we find

$$-\frac{\wp(z)}{(2\pi i)^2} = J_1^\bullet - \frac{1}{12}E_2.$$

We have one more relation to another classical function. The Jacobi theta function θ_1 is defined by

$$\theta_1(z, \tau) = -i \sum_{n \in \mathbb{Z}} (-1)^n p^{n+1/2} q^{\frac{(n+1/2)^2}{2}} = -iq^{1/8} (p^{1/2} - p^{-1/2}) \prod_{m \geq 1} (1 - q^m)(1 - pq^m)(1 - p^{-1}q^m)$$

where the last term is the Jacobi Triple product. Taking the logarithmic derivative we get

$$\frac{\theta_1^\bullet}{\theta_1} = J_1$$

Proposition 3. K_n satisfies $K_n(z/\tau, -1/\tau) = \tau^n K_n(z, \tau)$

Proof. Calculation! □

Next we like to consider the poles of J_n and K_n . From Remark 2 and the fact that θ_1 has a simple zero at $z = 0$ and nowhere else in the fundamental region $\{\lambda + \mu\tau \mid 0 \leq \lambda, \mu < 1\}$, we see that J_1 has a simple pole at $z = 0$ while no other singularities in the fundamental region. From the differential relation between the J_n - equation (2) - it now follows directly that J_n doesn't have any poles at all in the Fundamental region for $n \geq 3$. Note that this doesn't imply that J_n does not have poles at all, as can be seen by the periodicity relations.

We now turn to K_n . Consider first the Taylor expansion in z of a meromorphic function $\phi(z, \tau)$ that satisfies:

$$\phi(z, \tau + 1) = \phi(z, \tau), \quad \phi(z/\tau, -1/\tau) = \tau^m \phi(z, \tau)$$

The Laurent series of ϕ at $z = 0$ then reads

$$\phi(z, \tau) = \sum_{k \geq -N} (2\pi i z)^k a_k(\tau)$$

for some integer N , and some functions $a_k(\tau)$. From

$$\phi(z/\tau, -1/\tau) = \sum_{k \geq -N} \tau^{-k} (2\pi i z)^k a_k(-1/\tau) = \tau^m \phi(z, \tau)$$

we get that $a_k(-1/\tau) = \tau^{m+k} a_k(\tau)$. while from $\phi(z, \tau + 1) = \phi(z, \tau)$ we derive $a_k(\tau + 1) = a_k(\tau)$.

Take now K_n . Observe first that all Taylor coefficients of $K_n(z, \tau)$ are holomorphic in τ as can be seen from taking z to be fixed $\neq 0$. Lets now again fix $z \neq 0$ and take the limit $\tau \rightarrow i\infty$. It is clear from the Fourier expansions of $J_n, n \geq 1$ that all Taylor coefficients of $K_n(z, \tau)$ are bounded for $\tau \rightarrow \infty$. Hence we derived the following properties of K_n :

- K_n is double periodic, i.e. $K_n(z + \lambda\tau + \mu, \tau) = K_n(z, \tau)$
- K_n transforms like a weight n Jacobi Form, i.e. $K_n(z/\tau, -1/\tau) = \tau^n K_n(z, \tau)$
- K_n has a pole of order n at $z = 0$ (as J_1 has a pole of order 1). K_n has no other poles in the fundamental region.
- K_n has a Taylor expansion given by

$$K_n(z, \tau) = \sum_{k \geq -n} a_k(\tau) w^k$$

where a_k is a holomorphic modular form of weight $n + k$.

- $a_{-n} = (-1)^{n+1}(n - 1)$ so that

$$K_n(z, \tau) = \frac{(-1)^{n+1}(n - 1)}{w^n} + O(w^{-n+4}).$$

This can be seen from the Taylor expansion for J_1 above, the expression for K_n in terms of the J_i and the fact that the dimension of the space of Modular forms of weight k is 0 for $k = 1, 2, 3$.

- As K_n is an elliptic function with only pole at $z = 0$ the residuum at $z = 0$ satisfies $a_{-1} = 0$.

Corollary 3. *The principal part - der Hauptteil - of K_2, K_3, K_4, K_5 is given by $(-1)^{n+1}(n-1)\frac{1}{w^n}$ or*

$$K_n = \frac{(-1)^{n+1}(n-1)}{w^n} + O(const)$$

This will have an immediate consequence. Let \mathbb{V} be the \mathbb{C} -vector space given by all meromorphic functions

$$f : \mathbb{C} \times \mathbb{H} \rightarrow \mathbb{C} \cup \{\infty\}$$

such that f is a meromorphic Jacobi Form of index 0 with pole in the fundamental region only at $z = 0$ and such that f has a Laurent Series in the variable z at $z = 0$ with coefficients holomorphic modular forms (in τ).

Lemma 4. *\mathbb{V} is a module over the ring of holomorphic modular forms M_* and a basis over M_* is given by the functions K_i .*

Proof. This follows from the list of properties of K_n above and the fact that there is no elliptic function with a single pole of order 1. \square

A filtration of the module is given by

$$\mathbb{V}_n = \{f \in \mathbb{V} \mid f \text{ has a pole of order } \leq n\}$$

2.4 Differential operators

Recall what we call a meromorphic Jacobi Form. Let ϕ be a meromorphic Jacobi form of weight k and index 0. The elliptic equation then reduces for ϕ to the condition that ϕ is double periodic in z . Its immediate that ϕ^\bullet is again a meromorphic Jacobi form of weight $k+1$ and index 0. Additionally we define a q -derivative by

$$D_q \phi = \phi' - \phi^\bullet J_1 - \frac{k}{12} E_2 \phi$$

Proposition 4. *$D_q \phi$ is a meromorphic Jacobi form of weight $k+2$ and index 0.*

Proof. A simple check. \square

Note that the operator D_q sends \mathbb{V}_n to \mathbb{V}_{n+2} .

2.5 Relations

Given two meromorphic Jacobi forms of index 0 and the same principal part of their singularities, then they are equal up to a function of τ . Since for $n \leq 5$ we only have a single negative term in the Taylor expansion for K_n , we will have relations among products and derivatives of the K_i for low i . For products one may find

$$K_2 \cdot K_2 = -\frac{1}{3}K_4 + \frac{1}{60}E_4 \tag{4}$$

$$K_2 \cdot K_3 = -\frac{1}{2}K_5 \tag{5}$$

Similarly for derivatives, its easy to show the following identities coming from the differential operators defined previously.

$$\begin{aligned}
J_1^\bullet &= K_2 + \frac{1}{12}E_2(\tau) \\
K_2^\bullet &= K_3 \\
K_3^\bullet &= 2K_4 - \frac{7}{120}E_4 \\
K_4^\bullet &= 3K_5 \\
D_q K_2 &= \frac{2}{3}K_4 - \frac{1}{180}E_4 \\
D_q K_3 &= \frac{3}{2}K_5
\end{aligned}$$

Writing these equations in terms of the J_i we can get expressions for $J_i^\bullet, i = 1, 2, 3, 4$ in terms of J_i . Similarly, the equations (4) give relations among the J_i . So we have for example:

$$\begin{aligned}
J_2^2 &= -\frac{1}{3}J_4 + \frac{4}{3}J_3J_1 + \frac{1}{60}E_4 \\
J_1^\bullet &= J_2 - J_1^2 + \frac{1}{12}E_2 \\
J_2^\bullet &= J_3 - J_1J_2 + \frac{1}{6}E_2J_1 \\
J_3^\bullet &= 2J_4 - 5J_3J_1 + 3J_2^2 + \frac{1}{4}J_2E_2 - \frac{7}{120}E_4 \\
&= J_4 - J_3J_1 + \frac{1}{4}J_2E_2 - \frac{1}{120}E_4
\end{aligned}$$

Remark 5. We see from remark 2 that the Weierstrass \wp -function satisfies

$$-\frac{\wp(z)}{(2\pi i)^2} = J_1^\bullet - \frac{1}{12}E_2 = K_2$$

The classical differential equation for \wp , namely $\left(\frac{\partial}{\partial z}\wp(z)\right)^2 = 4\wp^3(z) - 60G_4\wp(z) - 140G_6$ then reads

$$-(K_2^\bullet)^2 = 4K_2^3 - \frac{1}{12}E_4K_2 - \frac{1}{216}E_6$$

3 Θ

The deformed Eisenstein Series J_n are closely related to the classical θ -Functions, a fact that we will show in this chapter. But first lets fix the notation.

3.1 Definitions

We use the usual, that means, the Wikipedia [W] definitions for the classical Jacobi theta Functions, which are:

$$\begin{aligned}\theta_1(z, \tau) &= -i \sum_{n \in \mathbb{Z}} (-1)^n p^{n+1/2} q^{\frac{(n+1/2)^2}{2}} = -i q^{1/8} p^{1/2} \theta_3(z + \frac{1}{2}\tau + \frac{1}{2}, \tau) \\ \theta_2(z, \tau) &= \sum_{n \in \mathbb{Z}} p^{n+1/2} q^{\frac{(n+1/2)^2}{2}} = q^{1/8} p^{1/2} \theta_3(z + \frac{1}{2}\tau, \tau) \\ \theta_3(z, \tau) &= \sum_{n \in \mathbb{Z}} p^n q^{\frac{n^2}{2}} \\ \theta_4(z, \tau) &= \sum_{n \in \mathbb{Z}} (-1)^n p^n q^{\frac{n^2}{2}} = \theta_3(z + \frac{1}{2}, \tau)\end{aligned}$$

We will write $\theta_i^{k\bullet}$ (resp. $\theta_i^{k'}$) for the k 'th derivative of θ_i with respect to z (resp. τ).

3.2 Relation to J_i

Define functions $g_m(\tau), m \geq 0$ by

$$\frac{1}{\theta_1(z, \tau)} = \sum_{m \geq 0} g_m(\tau) w^{2m-1}$$

where again $w = 2\pi iz$. Next set

$$h_n := \begin{cases} (2m)! g_m(\tau) \theta_1^\bullet(0) & \text{for } n = 2m \geq 0 \\ 0 & \text{for } n \text{ odd} \end{cases}$$

where we use the convention $0! = 1$. In particular $h_0 = 1$.

For $n \geq 0$ set

$$F_n(z, \tau) = \frac{1}{\theta_1} \left(\sum_{k=0}^n \binom{n}{k} h_{n-k} \theta_1^{k\bullet} \right)$$

Theorem 6.

$$F_n = J_n \quad \text{for all } n \geq 0.$$

Proof. Differentiating the equation

$$\theta_1(z + \lambda\tau, \tau) = -e^{-2\pi i(\lambda z + \frac{1}{2}\lambda^2\tau)} \theta_1(z, \tau)$$

we find

$$\theta_1^{k\bullet}(z + \lambda\tau, \tau) = - \sum_{l=0}^k (-1)^{l+k} \binom{k}{l} e^{-2\pi i(\lambda z + \frac{1}{2}\lambda^2\tau)} \lambda^{k-l} \theta_1^{l\bullet}$$

From this we find independent of h_k

$$\begin{aligned}F_n(z + \lambda\tau) &= \frac{1}{\theta_1} \left(\sum_{k=0}^n \binom{n}{k} h_{n-k} \sum_{l=0}^k (-1)^{l+k} \binom{k}{l} \lambda^{k-l} \theta_1^{l\bullet} \right) \\ &= \frac{1}{\theta_1} \left(\sum_{k=0}^n \sum_{l=0}^k \binom{n}{n-k+l} \binom{n-k+l}{l} (-1)^{n-k} h_{k-l} \lambda^{n-k} \theta_1^{l\bullet} \right) \\ &= \frac{1}{\theta_1} \left(\sum_{k=0}^n (-1)^{n+k} \lambda^{n-k} \binom{n}{k} \sum_{l=0}^k \binom{k}{l} h_{k-l} \theta_1^{l\bullet} \right) \\ &= \sum_{k=0}^n (-1)^{n+k} \binom{n}{k} \lambda^{n-k} F_k\end{aligned}$$

where we use the identity $\binom{n}{n-k+l}\binom{n-k+l}{l} = \binom{n}{k}\binom{k}{l}$.

Come so far, we use induction on n . For $n = 0$ nothing is to prove, $n = 1$ follows from taking the logarithmic derivative of the Jacobi Triple Product:

$$J_1 = \frac{\theta_1^\bullet}{\theta_1} = F_1$$

From now on assume that the statement is true for all $k < n$ where $n \geq 2$. Define for the scope of this proof the functions

$$\widetilde{K}_n = \sum_{k=0}^n (-1)^{n+k} \binom{n}{k} F_k F_1^{n-k}$$

The same recursion relation as for K_n is also true for \widetilde{K}_n .

Lets assume now that $n = 2m$ even. Its easy to see that $F_{2m}(z)$ does not have any poles for $z \in \{\lambda + \mu\tau \mid 0 \leq \lambda, \mu < 1\}$. Indeed, $\theta_1^{2k\bullet} = 2^k \theta_1^{k'}$ has a zero of order 1 at $z = 0$ hence $\frac{\theta_1^{2k\bullet}}{\theta_1}$ extends to a holomorphic function at $z = 0$. By induction we conclude that the principal part of \widetilde{K}_n is equal to the principal part of K_n . Therefor to prove that $F_{2m} = J_{2m}$ its left to show that $F_{2m}(0) = J_{2m}(0) = B_{2m}E_{2m}$. This is equivalent to the following identity:

$$\sum_{k=0}^m \binom{2m}{2k} h_{2m-2k} \frac{\theta_1^{2k\bullet}}{\theta_1}(0) = B_{2m}E_{2m}.$$

$\frac{\theta_1^{2k\bullet}}{\theta_1}$ is an even holomorphic function, hence

$$O(z) = \left(\frac{\theta_1^{2k\bullet}}{\theta_1}\right)^\bullet = \frac{\theta_1^{(2k+1)\bullet}}{\theta_1} - \frac{\theta_1^{2k\bullet}}{\theta_1} \frac{\theta_1^\bullet}{\theta_1}$$

Using $\frac{\theta_1^\bullet}{\theta_1} = J_1 = \frac{1}{w} + O(w)$, we obtain by comparing poles:

$$Res_{w=0} \left(\frac{\theta_1^{(2k+1)\bullet}}{\theta_1} \right) = \frac{\theta_1^{2k\bullet}}{\theta_1}(0).$$

The Taylor expansion of θ_1 is given by

$$\sum_{k \geq 0} \frac{\theta_1^{(2k+1)\bullet}(0)}{(2k+1)!} w^{2k+1}$$

We get $\theta_1 = w\theta_1^\bullet(0) + O(w^3)$ and hence

$$Res_{w=0} \left(\frac{\theta_1^{(2k+1)\bullet}}{\theta_1} \right) = \frac{\theta_1^{(2k+1)\bullet}(0)}{\theta_1^\bullet(0)}$$

Hence the equation that we need to prove is just

$$\sum_{k=0}^m \binom{2m}{2k} (2m-2k)! g_{m-k} \theta_1^{(2k+1)\bullet}(0) = B_{2m}E_{2m}.$$

or

$$\sum_{k=0}^m g_{n-k} \frac{\theta_1^{(2k+1)\bullet}(0)}{(2k)!} = \frac{B_{2m} E_{2m}}{(2m)!}.$$

which is equivalent to the statement that the $2m - 1$ -th Taylor coefficient of the left and right hand side of:

$$\frac{1}{\theta_1(z, \tau)} \cdot \theta_1(z, \tau)^\bullet = J_1$$

are the same.

Assume $n \geq 3$ is odd. As before its enough to show that $F_n = F_{2m+1}$ has no poles, and that $F_n(0) = 0$. The statement for the poles is equivalent to

$$\sum_{k=0}^m \binom{2m+1}{2k+1} h_{2m-2k} \text{Res}_{w=0} \left(\frac{\theta_1^{(2k+1)\bullet}}{\theta_1} \right) = 0$$

As before this is equivalent to

$$\sum_{k=0}^m g_{m-k} \frac{\theta_1^{(2k+1)\bullet}(0)}{(2k+1)!} = 0$$

which is again just the equality of the $2m$ -th coefficient of the Taylor expansion of the left and right hand side of

$$\frac{1}{\theta_1} \theta_1 = 1.$$

To show $F_{2m+1}(0) = 0$ we note that F_{2m+1} is an odd function. □

The above proof gives us a (recursive) expression of the h_k function in terms of Eisenstein Series. The equation

$$\sum_{k=0}^m \binom{2m+1}{2k+1} h_{2m-2k} \text{Res}_{w=0} \left(\frac{\theta_1^{(2k+1)\bullet}}{\theta_1} \right) = 0$$

we can rewrite into the form

$$(2m+1)h_{2m} + \sum_{k=1}^m \binom{2m+1}{2k+1} h_{2m-2k} 2^k \frac{\theta_1^{k'}}{\theta_1}(0) = 0$$

Let $P_k = \frac{\theta_1^{k'}}{\theta_1}(0)$. From again taking the logarithmic derivative with respect to τ of the Jacobi Triple product, we have $P_1 = \frac{1}{8}E_2$ and from differentiating we obtain the recursive formula:

$$P_{n+1} = P'_n + \frac{1}{8}E_2 P_n.$$

Out of this we can recursively obtain the h_k . For example

$$\begin{aligned} h_0 &= 1 \\ h_2 &= -\frac{1}{12}E_2 \\ h_4 &= -\frac{1}{10}(E_2' - \frac{7}{24}E_2^2) \\ h_6 &= -\frac{1}{7}(E_2'' - \frac{11}{8}E_2 E_2' + \frac{31}{192}E_2^3) \end{aligned}$$

And so we obtain the representations for J_i in terms of θ -functions, for example:

$$\begin{aligned}
J_0 &= 1 \\
J_1 &= \frac{\theta_1^\bullet}{\theta_1} \\
J_2 &= \frac{1}{\theta_1} \left(\theta_1^{\bullet\bullet} - \frac{1}{12} E_2 \theta_1 \right) \\
J_3 &= \frac{1}{\theta_1} \left(\theta_1^{\bullet\bullet\bullet} - \frac{1}{4} E_2 \theta_1^\bullet \right) \\
J_4 &= \frac{1}{\theta_1} \left(\theta_1^{\bullet\bullet\bullet\bullet} - \frac{1}{2} E_2 \theta_1^{\bullet\bullet} + \left(-\frac{1}{10} E_2' + \frac{7}{240} E_2^2 \right) \theta_1 \right) \\
J_5 &= \frac{1}{\theta_1} \left(\theta_1^{5\bullet} - \frac{5}{6} \theta_1^{\bullet\bullet\bullet} + \left(-\frac{1}{2} E_2' + \frac{7}{48} E_2^2 \right) \theta_1^\bullet \right)
\end{aligned}$$

Lets describe these relation in a more compact form. Define the generating series

$$\mathcal{J} = \sum_{n \geq 0} \frac{J_n(z, \tau)}{n!} x^n$$

for a variable x . Then by definition of the h_n we have

$$\frac{x \theta_1^\bullet(0)}{\theta_1(\frac{x}{2\pi i}, \tau)} = \sum_{m \geq 0} \frac{h_{2m}(\tau)}{(2m)!} x^{2m}$$

Applying Theorem 6 to \mathcal{J} we get

Corollary 7.

$$\mathcal{J} = x \theta_1^\bullet(0, \tau) \frac{\exp(x \partial_p) \cdot \theta_1(z, \tau)}{\theta_1(z, \tau) \theta_1(\frac{x}{2\pi i}, \tau)}$$

with

$$\exp(x \partial_p) \cdot \theta_1(z, \tau) := \sum_{k \geq 0} \frac{x^k}{k!} \theta_1^{k\bullet}$$

3.3 Consequences

As noted in the beginning, the J_i functions satisfy the following easy set of differential equations:

$$\frac{1}{k} J_k^\bullet = \frac{1}{k-1} J_{k-1}' \quad (6)$$

When we plug in F_k for J_k , we get a differential equation for θ_1 . When $k = 2$, the equation will be trivial, but for $k = 3$ we get

Corollary 8.

$$\theta_1^{\bullet\bullet\bullet\bullet} \theta_1 - 4 \theta_1^{\bullet\bullet\bullet} \theta_1^\bullet + 3 \theta_1^{\bullet\bullet} \theta_1^{\bullet\bullet} - \theta_1 \theta_1^{\bullet\bullet} E_2 + (\theta_1^\bullet)^2 E_2 + \frac{1}{2} \theta_1^2 E_2' = 0 \quad (7)$$

When using 6 for the function \mathcal{J} , we get

Corollary 9.

$$\mathcal{J}' = \left(\frac{\partial}{\partial x} - \frac{1}{x} \right) \mathcal{J}$$

3.4 The other theta functions

In this section we follow the lines above and show that basically most of the statements done so far are also true for $\theta_2, \theta_3, \theta_4$. Recall

$$\begin{aligned}\theta_2(z, \tau) &= \frac{\eta(q)^2}{\eta(q^2)} \frac{\theta_1(2z, 2\tau)}{\theta_1(z, \tau)} = \theta_1\left(z + \frac{1}{2}, \tau\right) \\ \theta_3(z, \tau) &= q^{1/8} p^{1/2} \theta_1\left(z + \frac{1}{2}\tau + \frac{1}{2}, \tau\right) \\ \theta_4(z, \tau) &= \theta_3(z + 1/2, \tau) = -iq^{1/8} p^{1/2} \theta_1\left(z + 1/2\tau, \tau\right)\end{aligned}$$

By using the Jacobi product formula we can find:

$$\begin{aligned}\frac{\theta_2^\bullet}{\theta_2} &= 2J_1(2z, 2\tau) - J_1(z, \tau) \\ \frac{\theta_3^\bullet}{\theta_3} &= 2J_1(2z, \tau) - 2J_1(2z, 2\tau) + J_1(z, \tau) - J_1(z, \tau/2) \\ \frac{\theta_4^\bullet}{\theta_4} &= J_1(z, \tau/2) - J_1(z, \tau)\end{aligned}$$

And so we define

$$\begin{aligned}J_{1,n} &= J_n \\ J_{2,n} &= 2J_n(2z, 2\tau) - J_n(z, \tau) \\ J_{3,n} &= 2^{2-n} J_n(2z, \tau) - 2J_n(2z, 2\tau) + J_n(z, \tau) - 2^{1-n} J_n(z, \tau/2) \\ J_{4,n} &= \frac{1}{2^{n-1}} J_n(z, \tau/2) - J_n(z, \tau)\end{aligned}$$

This definition is made so that the functions $J_{i,n}$ satisfy for $k \geq 1$ the differential relation

$$\frac{k}{k+1} J_{i,k+1}^\bullet = J_{i,k}' \quad (8)$$

It is easy to check that

$$J_{i,n}(z + \lambda\tau + \mu, \tau) = \sum_{k=0}^n (-1)^{n+k} \binom{n}{k} \lambda^{n-k} J_{i,k}(z, \tau)$$

Also it follows directly that for $i = 2, 3, 4$ we have

$$J_{i,n}(z/\tau, -1/\tau) = \sum_{k=0}^n \binom{n}{k} z^{n-k} \tau^k J_{6-i,k}$$

So we can define as before

$$K_{i,n} = \sum_{k=0}^n (-1)^{n+k} \binom{n}{k} J_{i,k} J_{i,1}^{n-k}$$

and as above we get that $K_{i,n}$ is an elliptic function with transformation property:

$$K_{i,n}(z/\tau, -1/\tau) = K_{6-i,n}(z, \tau)$$

for $i = 2, 3, 4$. To give a more concrete description a calculation shows

$$\begin{aligned} J_{2,n}(z, \tau) &= \delta_{n,1} \frac{p}{p+1} + B_n - n \sum_{k,r \geq 1} (-1)^k r^{n-1} (p^k + p^{-k}) q^{kr} \\ J_{3,n}(z, \tau) &= -B_n \left(1 - \frac{1}{2^{n-1}}\right) - n \sum_{k,r \geq 1} \left(r - \frac{1}{2}\right)^{n-1} (-1)^k (p^k + (-1)^n p^{-k}) q^{k(r-\frac{1}{2})} \\ J_{4,n}(z, \tau) &= -B_n \left(1 - \frac{1}{2^{n-1}}\right) - n \sum_{k,r \geq 1} \left(r - \frac{1}{2}\right)^{n-1} (p^k + (-1)^n p^{-k}) q^{k(r-\frac{1}{2})} \end{aligned}$$

From this description we find:

$$\begin{aligned} J_{2,n}(z, \tau) &= J_{1,n}\left(z + \frac{1}{2}\right) \\ J_{3,n}(z, \tau) &= \sum_{l=0}^n \binom{n}{l} \frac{1}{2^{n-l}} J_l\left(z + \frac{1}{2} + \frac{1}{2}\tau\right) \\ J_{4,n}(z, \tau) &= \sum_{l=0}^n \binom{n}{l} \frac{1}{2^{n-l}} J_l\left(z + \frac{1}{2}\tau\right) \end{aligned}$$

In this calculation one uses the identity

$$B_n \left(-1 + \frac{1}{2^{n-1}}\right) - \sum_{l=0}^n \binom{n}{l} \frac{1}{2^{n-l}} B_l = 0$$

which can be derived by a three step affair: First prove

$$-\frac{x}{e^x - 1} + 2 \frac{x/2}{e^{x/2} - 1} = e^{x/2} \frac{x}{e^x - 1}$$

then plugin the Taylor expansion

$$-\sum_{n \geq 0} \frac{B_n}{n!} x^n + 2 \sum_{n \geq 0} \frac{1}{2^n} \frac{B_n}{n!} x^n = \left(\sum_{k \geq 0} \frac{x^k}{2^k k!}\right) \left(\sum_{l \geq 0} \frac{B_l}{l!} x^l\right)$$

and at last expand and compare coefficients.

We can now prove

Theorem 10. *Let $h_k, k \geq 0$ as before. We have*

$$J_{i,n} = \frac{1}{\theta_i} \left(\sum_{k=0}^n \binom{n}{k} h_{n-k} \theta_i^{k \bullet} \right)$$

for $i = 1, 2, 3, 4$.

Proof. $i = 1$ has already been proven. For $i = 2$, note that

$$\begin{aligned} J_{2,n}(z, \tau) = J_{1,n}(z + \frac{1}{2}) &= \sum_{k=0}^n \binom{n}{k} h_{n-k} \frac{\theta_1^{k\bullet}}{\theta_1}(z + \frac{1}{2}) \\ &= \sum_{k=0}^n \binom{n}{k} h_{n-k} \frac{\theta_1(z + \frac{1}{2})^{k\bullet}}{\theta_1(z + \frac{1}{2})} \\ &= \sum_{k=0}^n \binom{n}{k} h_{n-k} \frac{\theta_2(z)^{k\bullet}}{\theta_2(z)} \end{aligned}$$

For $i = 3$, note first that

$$\frac{\theta_3^{j\bullet}}{\theta_3} = \sum_{k=0}^j \binom{j}{k} \frac{1}{2^{j-k}} \frac{\theta_1^{k\bullet}}{\theta_1}(z + \frac{1}{2} + \frac{1}{2}\tau)$$

to find

$$\begin{aligned} J_{3,n} &= \sum_{l=0}^n \binom{n}{l} \frac{1}{2^{n-l}} J_l(z + \frac{1}{2} + \frac{1}{2}\tau) \\ &= \sum_{l=0}^n \binom{n}{l} \frac{1}{2^{n-l}} \sum_{k=0}^l h_{l-k} \binom{l}{k} \frac{\theta_1^{k\bullet}}{\theta_1}(z + \frac{1}{2} + \frac{1}{2}\tau) \\ &= \sum_{k=0}^n \sum_{l=k}^n \binom{n}{l} \binom{l}{k} \frac{1}{2^{n-l}} h_{l-k} \frac{\theta_1^{k\bullet}}{\theta_1}(z + \frac{1}{2} + \frac{1}{2}\tau) \\ &= \sum_{k=0}^n \sum_{j=k}^n \binom{n}{n-j+k} \binom{n-j+k}{k} \frac{1}{2^{j-k}} h_{n-j} \frac{\theta_1^{k\bullet}}{\theta_1}(z + \frac{1}{2} + \frac{1}{2}\tau) \\ &= \sum_{j=0}^n \binom{n}{j} h_{n-j} \sum_{k=0}^j \binom{j}{k} \frac{1}{2^{j-k}} \frac{\theta_1^{k\bullet}}{\theta_1}(z + \frac{1}{2} + \frac{1}{2}\tau) \\ &= \sum_{j=0}^n \binom{n}{j} h_{n-j} \frac{\theta_3^{j\bullet}}{\theta_3} \end{aligned}$$

as claimed. For $i = 4$ note that $\theta_4(z, \tau) = \theta_3(z + \frac{1}{2}, \tau)$ and so a similar argument as for $i = 2$ applies here. \square

For $i = 1, 2, 3, 4$ define the function

$$\mathcal{J}_i = \sum_{n \geq 0} \frac{J_{i,n}(z, \tau)}{n!} x^n$$

then (8) und Theorem 10 give us

Corollary 11.

$$\begin{aligned} \mathcal{J}_i &= x \frac{\theta_1^{i\bullet}(0, \tau)}{\theta_1(\frac{x}{2\pi i})} \frac{\exp(x\partial_p) \cdot \theta_i(z, \tau)}{\theta_i(z, \tau)} \\ \mathcal{J}'_i &= (\partial_x - \frac{1}{x}) \mathcal{J}_i^{i\bullet} \end{aligned}$$

Corollary 12. *The equation (7) above holds also when θ_1 is replaced by $\theta_2, \theta_3, \theta_4$.*

4 ϕ

4.1 Definition

Lets recall from [EZ85] some Jacobi forms. There are the Eisenstein series of weight 4,6 and index 1 called $E_{4,1}, E_{6,1}$. They define the two cusp forms

$$\begin{aligned}\phi_{10,1} &= \frac{1}{144}(E_6 E_{4,1} - E_4 E_{6,1}) \\ \phi_{12,1} &= \frac{1}{144}(E_4^2 E_{6,1} - E_6 E_{6,1})\end{aligned}$$

In this section we are mainly interested in

$$\phi := \phi_{-2,1} = \frac{\phi_{10,1}}{\Delta}$$

Here, Δ is the usual modular discriminant. There is an easy presentation for ϕ in terms of the θ_1 function. Recall from [DMZ12] that

$$\phi = \left(\frac{\theta_1}{\theta_1^\bullet(0)} \right)^2$$

Similarly one has

$$\begin{aligned}E_{4,1} &= \phi(-E_4 K_2 - \frac{1}{12} E_6) = \left(\frac{\theta_1}{\theta_1^\bullet(0)} \right)^2 \cdot \left(E_4 \frac{\wp(z)}{(2\pi i)^2} - \frac{1}{12} E_6 \right) \\ E_{6,1} &= \phi(-E_6 K_2 - \frac{1}{12} E_4^2) = \left(\frac{\theta_1}{\theta_1^\bullet(0)} \right)^2 \cdot \left(E_6 \frac{\wp(z)}{(2\pi i)^2} - \frac{1}{12} E_4^2 \right) \\ \phi_{12,1} &= -\frac{3}{\pi^2} \wp(z) \phi_{10,1} = -12 K_2 \phi_{10,1} = -12 K_2 \Delta \phi\end{aligned}$$

4.2 Differential operators

Let F be a weak Jacobi form of weight k and index m in the sense of [DMZ12], we can consider the function $\frac{F}{\phi^m}$. This will be a meromorphic Jacobi Form of weight $k + 2m$ and index 0, in particular it will be double periodic. The only pole of F/ϕ^m in a fundamental region is at $z = 0$ of order $\leq 2m$. A (weak) Jacobi Form has a Taylor expansion with coefficients quasi modular forms [DMZ12] and so we find that $F/\phi^m \in \mathbb{V}_{2m}$ where \mathbb{V}_n was defined right after Corollary 3. Hence dividing by ϕ^m gives a map

$$\begin{aligned}\omega_m &: \tilde{J}_{*,m} \rightarrow \mathbb{V}_{2m} \\ F &\mapsto \frac{F}{\phi^m}\end{aligned}$$

Lemma 13. ω_m is an isomorphism of M_* -Modules.

As there are no elliptic functions with a single pole of order 1, it follows a result also found in [EZ85]:

Corollary 14 ([EZ85]). $\tilde{J}_{2k+1,1} = 0$

We have three kind of operators acting on \mathbb{V} , namely

- Multiplication by K_i :

$$\begin{array}{ccc} K_i \cdot & : & \mathbb{V}_n \rightarrow \mathbb{V}_{n+i} \\ f & \mapsto & K_i \cdot f \end{array}$$

- Differentiation by z :

$$D_p : \mathbb{V}_n \rightarrow \mathbb{V}_{n+1}, \quad f \mapsto f^\bullet$$

- Differentiation by τ via D_q (Section 2.4)

$$D_q : \mathbb{V}_n \rightarrow \mathbb{V}_{n+2}, \quad f \mapsto f' - J_1 f^\bullet - \frac{k}{12} E_2 f$$

This gives operators on meromorphic Jacobi of index m forms via ω_m , namely we define:

$$\begin{aligned} D_p F &:= \phi^m \left(\frac{F}{\phi^m} \right)^\bullet = F^\bullet - 2m J_1 F \\ D_q F &:= \phi^m D_q \left(\frac{F}{\phi^m} \right) = F' - J_1 F^\bullet + (-m J_2 + 2m J_1^2 - \frac{k}{12} E_2) F \end{aligned}$$

These two operators map holomorphic forms to forms with a pole at $z = 0$. This happens because when F/ϕ^m has a pole of order $2m$, $D_q(F/\phi^m)$ and $D_p(F/\phi^m)$ will have a pole of order $2m + 2$ and $2m + 1$ respectively. When multiplying with ϕ^m we remove only singularities of order less than $2m$ and so we obtain a pole.

To avoid this, we will combine linear combinations of the operators above to cancel the coefficients of poles higher than $2m$. In this way we obtain differential operators for weak Jacobi Forms. We illustrate this in degree 2.

Degree 2:

We have 3 different operators in degree 2, namely D_q, D_p^2 and multiplication by K_2 . Lets analyze their action on the monomials $\frac{1}{w^n}$ and $\frac{1}{w^{n-1}}$:

$$\begin{aligned} D_q \left(\frac{1}{w^n} \right) &= \frac{n}{w^{n+2}} + O(w^{-n}) & D_q \left(\frac{1}{w^{n-1}} \right) &= (n-1) \frac{1}{w^{n+1}} + O(w^{n-1}) \\ D_p^2 \left(\frac{1}{w^n} \right) &= n(n+1) \frac{1}{w^{n+2}} + O(w^{-n}) & D_p^2 \left(\frac{1}{w^{n-1}} \right) &= n(n-1) \frac{1}{w^{n+1}} + O(w^{n-1}) \\ K_2 \frac{1}{w^n} &= \frac{-1}{w^{n+2}} + O(w^{-n}) & K_2 \frac{1}{w^{n-1}} &= -\frac{1}{w^{n+1}} + O(w^{n-1}) \end{aligned}$$

There are two possible combinations to cancel the $\frac{1}{w^{n+2}}$ term. Define

$$\begin{aligned} Heat &= 2n D_q - D_p^2 + n(n-1) K_2 \cdot \\ T_q &= D_q + n K_2 \end{aligned}$$

From those two only the *Heat* operator satisfies $Heat(\frac{1}{w^{n-1}}) = O(w^{n-1})$. We see that *Heat* is the unique linear combination of D_q, D_p^2, K_2 that sends $\mathbb{V}_n \rightarrow \mathbb{V}_n$.

Lets assume $n = 2m$ and set

$$\mathbb{V}_{2m}^{even} = \{f \in \mathbb{V}_{2m} \mid f \text{ is even}\}.$$

We have $T_q : V_{2m}^{even} \rightarrow V_{2m}^{even}$ and

$$\mathbb{V}_{2m}^{even} \cong \tilde{J}_{2*,m}$$

where the right hand side is the space of even weight weak Jacobi Forms. We define now:

Definition 15. Let $F \in \tilde{J}_{k,m}$, set

$$\begin{aligned} Heat(F) &= (4mD_q - D_p^2 + 2m(2m-1)K_2 \cdot)F = 4mF' - F^{\bullet\bullet} - \frac{1}{3}m(k - \frac{1}{2})E_2F \\ T_q(F) &= (D_q + 2mK_2)F = F' - J_1F^\bullet + (mJ_2 - \frac{k}{12}E_2)F \end{aligned}$$

It follows that $Heat$ acts on weak Jacobi Forms while T_q acts on even weight weak Jacobi Forms. It is easy to see that $Heat$ sends Jacobi Forms to Jacobi Forms ([DMZ12]). The $Heat$ operator has been studied for a while and appears in several other places, e.g. [EZ85], [DMZ12], [GK09]. On the other hand, T_q preserves holomorphicity only for even weight forms, but in this case it also preserves the Jacobi Forms, so that we got:

Theorem 16. T_q sends Jacobi forms of weight $2k$, index m to Jacobi Forms of weight $2k+2$, index m .

Proof. Let F be an even weight Jacobi form of weight $2k$ and index m and denote $\tilde{F} = \frac{F}{\phi^m}$. From the above we deduce that $T_q F$ is a holomorphic function and also that it satisfies the elliptic and modular transformation equations. We need to prove that $T_q F$ has a Fourier expansion of the form

$$\sum_{n \geq 0} \sum_{\substack{r \in \mathbb{Z} \\ r^2 \leq 4nm}} c(n, r) p^r q^n$$

This is equivalent [DMZ12] to proving that $\forall \alpha, \beta \in \mathbb{Q}$,

$$q^{m\alpha^2} T_q(F)(\alpha\tau + \beta, \tau) \text{ is bounded for } \tau \rightarrow \infty$$

We split this into two cases.

Case A ($\alpha \in \mathbb{Q} \setminus \mathbb{Z}$ or $\beta \in \mathbb{Q} \setminus \mathbb{Z}$): In this case $\theta_1(\alpha\tau + \beta, \tau) \neq 0$. Since

$$q^{\alpha^2/2} \theta_1(\alpha\tau + \beta, \tau) = -i \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2}(n + \frac{1}{2})^2}$$

let $\gamma \in \mathbb{Q}^{\geq 0}$ s.t. $q^{\alpha^2/2} \theta_1(\alpha\tau + \beta, \tau) \approx q^\gamma$ as $q \rightarrow 0$. Also we have that $\theta_1^\bullet(0, \tau) \approx q^{1/8}$ as $\tau \rightarrow \infty$. Since F is a Jacobi Form

$$q^{m\alpha^2} F(\alpha\tau + \beta, \tau) \approx q^\delta \text{ for } \tau \rightarrow \infty$$

for some $\delta \geq 0$. We see that

$$\frac{F}{\phi^m}(\alpha\tau + \beta, \tau) \approx q^{\frac{m}{4} + \delta - 2m\gamma} \text{ for } \tau \rightarrow \infty$$

Taking derivative with respect to z or τ might only increase the order of convergence so that $\left(\frac{F}{\phi^m}\right)^\bullet$ and $\left(\frac{F}{\phi^m}\right)'$ are also of the same or higher order of convergence when plugging in $z = \alpha\tau + \beta$ and letting $\tau \rightarrow \infty$. Since $\frac{F}{\phi^m}$ is double periodic we can further restrict to the case $0 \leq \alpha, \beta < 1$ (where $\alpha, \beta = 0$ is excluded). It is now enough to show that $J_i(\alpha\tau + \beta, \tau)$ is

bounded when $\tau \rightarrow \infty$. The result then follows, since the operator T_q is build out of derivatives with respect to z , τ and multiplication by J_i so that $T_q(\frac{F}{\phi^m})(\alpha\tau + \beta)$ has the same order of convergence as $\frac{F}{\phi^m}$ and so the result follows when multiplying with ϕ^m .

To see that $J_i(\alpha\tau + \beta, \tau)$ is bounded it is enough to consider the Fourier expansion and plug in $p = e^{2\pi i\beta} q^\alpha$ and check if all exponents of q are ≥ 0 . Since we restricted α to satisfy $0 \leq \alpha < 1$ this is immediate.

Case B ($\alpha, \beta \in \mathbb{Z}$): In this case, the general case $\alpha, \beta \in \mathbb{Z}$ follows directly from $\alpha = \beta = 0$ via the elliptic transformation equation that F satisfies. So we need to show that

$$T_q(F)(0, \tau) \text{ is bounded for } \tau \rightarrow \infty$$

Let $F = F_0 + w^2 F_2 + O(w^4)$, with F_0, F_2 quasi modular forms. Then

$$(T_q F)(0, \tau) = F'_0 + \left(\frac{m}{6} - \frac{k}{12}\right) E_2 F_0 - 2F_2$$

which is bounded for $\tau \rightarrow \infty$. □

From here we restrict our attention to even Jacobi Forms. T_q can be seen as a q -derivative for even weight Jacobi Forms. We will look at one more operator which we will call the generalized Serre derivative.

Definition 17. Let $F \in J_{k,m}$ with k even. Set

$$\begin{aligned} \partial^J F &= \partial_{k,m}^J F = \left(\frac{1}{1-4m} (T_q - \text{Heat}) \right) F \\ &= F' - \frac{k}{12} E_2 F + \frac{1}{1-4m} \left(F^{\bullet\bullet} - J_1 F^\bullet + m J_2 F - \frac{m}{6} E_2 F \right) \end{aligned}$$

From above we know that ∂^J sends even Jacobi Forms to even Jacobi Forms of weight $+2$. Let F have the Taylor expansion

$$F = F_0 + w^2 F_2 + O(w^4)$$

then a calculation shows

$$\partial^J F = \partial^{SD} F_0 + O(w^2)$$

where ∂^{SD} is the classical Serre derivative. Further note that $\partial_{k,0}^J = \partial^{SD}$ and so

$$\partial_{k,0}^J (F(0, \tau)) = (\partial_{k,m}^J F)(0, \tau)$$

So we see that ∂^J is a direct extension of the Serre derivative for modular Forms.

Degree 3: The method is the same. For degree 3 we have 4 operators $D_p D_q, D_p^3, K_2 D_p, K_3$. If we like to find an operator for weak Jacobi Forms we need to ensure that this operator sends \mathbb{V}_n to itself. This gives 3 linear conditions and we see that there is an unique differential operator for weak Jacobi Forms of degree 3 (given also in [GK09]). If we restrict ourself to even weight forms, we have to obey two linear condition so that there are 2 linearly independent operators for even weight weak Jacobi Forms. Note that each of these 2 operators annihilates all forms of index 1 (as $J_{2k+1,1} = 0 \forall k$).

Degree $n \geq 4$: Finding operators in this degree boils as above down to solving a system of linear equations.

4.3 Ramanujan's Equation for Jacobi Forms

Consider the presentations of $E_{4,1}$ and $E_{6,1}$

$$\begin{aligned} E_{4,1} &= \phi(-E_4 K_2 - \frac{1}{12} E_6) \\ E_{6,1} &= \phi(-E_6 K_2 - \frac{1}{12} E_4^2) \end{aligned}$$

Definition 18.

$$E_{2,1} := \phi(-E_2 K_2 - \frac{1}{12} E_4)$$

$E_{2,1}$ satisfies the following properties:

- (a) holomorphic on $\mathbb{C} \times \mathbb{H}$
- (b) has a Fourier expansion

$$E_{2,1}(z, \tau) = \sum_{n \geq 0} \sum_{\substack{r \in \mathbb{Z} \\ r^2 \leq 4n}} c(n, r) p^r q^n$$

In particular $c(n, r) = 0$ for $4n - r^2 < 0$.

- (c) satisfies the elliptic transformation equation, while the modular equation reads

$$E_{2,1}(z/\tau, -1/\tau) = e^{\frac{2\pi i z^2}{\tau}} \tau^2 E_{2,1} + \frac{1}{2\pi i} e^{\frac{2\pi i z^2}{\tau}} \tau \phi_{0,1}$$

- (d) $E_{2,1}(0, \tau) = E_2(\tau)$
- (e) The first Fourier coefficients $c(n, r)$ are given by

$\begin{smallmatrix} r \\ n \end{smallmatrix}$	-4	-3	-2	-1	0	1	2	3	4
0	0	0	0	0	1	0	0	0	0
1	0	0	1	-28	30	-28	1	0	0
2	0	0	30	-264	396	-264	30	0	0
3	0	-28	396	-1620	2408	-1620	396	-28	0
4	1	-264	2408	-7944	11430	-7944	2408	-264	1

Remark 19. *i) The conditions b) and d) above basically determine the function $E_{2,1}$ but a more conceptual definition is yet to be given.*

ii) In [Cho97] Choie introduced a function, which she called $E_{2,1}$. Her function is different from the one above.

Analogous to the Modular case we have now with a small calculation

Theorem 20 (Ramanujan Equations for index 1 Jacobi Forms).

$$\begin{aligned} \partial^J E_{2,1} + \frac{1}{12} E_2 E_{2,1} + \frac{1}{16} E_4' \phi_{-2,1} &= -\frac{1}{12} E_{4,1} \\ \partial^J E_{4,1} &= -\frac{1}{3} E_{6,1} \\ \partial^J E_{6,1} &= -\frac{1}{2} E_4 E_{4,1} \end{aligned}$$

This can be seen as a direct generalization of Ramanujan's original equations. Indeed it is easy to see that $E_{2k,1}(z=0) = E_{2k}$ for $k = 1, 2, 3$ and when restricting the above equations to $z = 0$ we get the equations (1).

4.4 Just for Fun

At last we want to give another presentation for the function $\phi = \phi_{-2,1}$. I like to thank Emanuel Scheidegger for pointing this connection out to me. We have

Theorem 21.

$$\phi_{-2,1} = \frac{-2}{\theta_4(0, 2\tau)^4} \left(\frac{\theta_3(2z, 2\tau)}{\theta_3(0, 2\tau)} - \frac{\theta_2(2z, 2\tau)}{\theta_2(0, 2\tau)} \right)$$

Proof. Both functions are even, vanish at $z = 0$ (to second order) and have the same periodicity factors, hence they are equal to a constant. We have

$$\phi^{\bullet\bullet}(0) = 2$$

and

$$RHS^{\bullet\bullet}(0) = \frac{1}{\theta_4(0, 2\tau)^4} \left(\frac{2}{3}E_2(\tau) - 4E_2(2\tau) + \frac{16}{3}E_2(4\tau) \right) = 2$$

and the result follows. □

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